Stable, Renormalizable, Scalar Tachyonic Quantum Field Theory with Chronology Protection

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Abstract

We use microlocal arguments to suggest that Lorentz symmetry breaking must occur in a reasonably behaved tachyonic quantum field theory that permits renormalizability. In view of this, we present a scalar tachyonic quantum field model with manifestly broken Lorentz symmetry and without exponentially growing/decaying modes. A notion of causality, in which anti-telephones are excluded, and which is viewed as a form of chronology protection, is obeyed. The field theory is constructed in a preferred tachyon frame in terms of commuting creation/annihilation operators. We calculate some sample (renormalized) operators in this preferred frame, argue that the Hadamard condition is satisfied, and discuss the PCT and spin-statistics theorems for this model.

1 Introduction

Ever since the notion of *tachyons*, i.e., particles which always travel faster than light, was conceived as a possibility within the basic framework of special relativity [2], a quantum field theory describing such particles has been sought. Despite numerous attempts to formulate such a theory [35, 10, 1, 8, 33], it appears that a model which is consistent with the desiderata of conventional quantum field theory has been wanting [20, 21, 18]. However, due to the interest in the possibility that the neutrino may turn out to be a tachyon [6], which has been supported, at least naïvely, by tritium beta decay experiments during the 90's [13], one perhaps should be motivated to clarify the issue either by presenting a viable model,

or by showing that no viable tachyonic quantum field model, according to some reasonable definition of "viable", can exist.

It is apparent from the early literature cited above, that various authors have attempted to hold tightly to the assumption of *Poincaré invariance*, no matter what the consequences for the theory. The usual way of requiring such invariance in the context of a scalar quantum field theory is by demanding that the *two-point distribution* $\langle 0 \,|\phi(x)\phi(y)|\, 0\rangle$ be left unchanged in value when the spacetime points x, y are simultaneously replaced by $\Lambda x + a, \Lambda y + a$ respectively. (Here, $|0\rangle$ is some notion of "vacuum" or "ground" state, $\phi(x)$ is an operator-valued distribution called the *field operator*, Λ is a proper, orthochronous Lorentz transformation, and a is a constant spacetime vector.) However, such a restriction, under further suitable physical assumptions, as will be explained in Section 2, must necessarily lead to a non-renormalizable theory, i.e., one which could not be incorporated into any renormalizable interacting (or self-interacting) theory, and for which the renormalized stress-energy tensor of the free field would make no sense. The argument used in Section 2 comes from the arena of *microlocal quantum field theory* [27, 29], which has proven to be reasonably successful, for example, in clarifying renormalization in *quantum field theory on curved spacetime* [4, 15].

From the standpoint of quantum field theory on curved spacetime, it is clear that, applying the usual methods to construct a scalar model of tachyons on flat *Minkowski* spacetime, one obtains a quantum field theory satisfying the usual axioms of *Wightman positivity* (namely the condition on the two-point distribution corresponding to the positive definiteness of the Hilbert space inner product), local commutativity, and the *Hadamard condition*, with, however, no guarantee of *Poincaré invariance.* (Note that this does not nullify the existence of Poincaré *symmetry*, which is always a symmetry of the underlying Minkowski spacetime, but that one allows for the possibility that the construction of the various quantities needed in the quantum field theory may manifest *spontaneous Poincaré symmetry breaking.*) That one is permitted to effect this construction, follows from the quite general existence (and

uniqueness) theorems for the typical quantities needed for a quantum field theory, such as the advanced and retarded fundamental solutions Δ_A, Δ_R to the inhomogeneous wave equation on a globally hyperbolic spacetime [26, 25]. Indeed, there is also an existence theorem (and, up to smooth function, a uniqueness theorem) for a Feynman propagator satisfying the Hadamard condition on such a spacetime [9, 27, 29]. Furthermore, the smooth part of the Feynman propagator may be chosen so that the corresponding two-point distribution satisfies Wightman positivity [9]. Note that it would be at the point of introducing this smooth function that the Poincaré symmetry may need to be broken.

Carrying out this construction for the *tachyonic* Klein-Gordon equation (i.e., with the opposite sign in the mass squared term) leads, however, to a theory which yields exponentially growing renormalized expectation values of typical observables, since in order to ensure that the strict local commutativity condition is satisfied, the mode solutions of the wave equation which are exponentially growing and decaying in time must be included in the mode expansion of the field operator. This situation is unpalatable from a physical point of view, since we would hope that a valid quantum field theory describing tachyonic neutrinos would at least preserve *stability.* To support this, one may simply observe that neutrinos from SN1987A, which were detected after travelling a distance of 150,000 light years at very close to the speed of light, have evidently manifested this property. A further aspect of the theory which may at first glance appear slightly disconcerting is that the two-point distribution does indeed manifest explicit breaking of Lorentz invariance *and* time translation invariance (but space translation invariance is maintained).

There is at our disposal, however, the simple procedure of deleting the undesirable exponentially growing/decaying modes from the theory. Since these modes contribute only a smooth function to the two-point distribution, the resulting theory still satisfies the Hadamard condition, and so retains the possibility of being incorporated into a renormalizable interacting or self-interacting quantum field theory according to the criteria of

Weinberg's theorem [4]. The theory also maintains Wightman positivity since it is explicitly obtained from a mode sum. Two new features (from the point of view of conventional QFT) remain or emerge (from the construction described in the previous paragraph): the theory still breaks Lorentz invariance (but now preserves both time and space translation invariance), and the usual notion of local commutativity (commuting fields at spacelike separation) is not satisfied.

However, neither of these two features appears insurmountable. Indeed, the notion of a preferred *tachyon frame*, "seen" only by the tachyons in the theory, would constitute one of the predictions of the theory. (Note particularly that the preferred tachyon frame is not one in which the speed of light takes on a special value c, different from what would be measured in other frames; rather c is the same in every inertial frame, as in the usual formulation of special relativity.) Furthermore, the local commutativity axiom is violated so weakly that physical signals, made up of tachyons described by this QFT, can be sent backward in time, but only to points which are spacelike separated from the sender. This is true even for devices which use relays to attempt to send messages backward in time to the sender (called *anti-telephones*). Such devices cannot be constructed, according to this QFT, since the (tachyonic) particles required for such a device cannot be simultaneously created from any of the vacuum states allowed in the theory. We consider this property to be a manifestation of *chronology protection*, which has otherwise appeared in the quite different context of QFT on curved spacetime [14, 22].

Working in the preferred tachyon frame, we begin the construction of the scalar model from scratch, presenting some of the Green's functions, in Section 3. Continuing in the preferred frame, we then utilize a Lagrangian approach to determine the (renormalized) Hamiltonian and momentum operators for a Hermitian scalar tachyon, as well as the charge operator for a charged scalar tachyon, in Section 4. Further elaboration on the Hadamard and chronology protection properties, as well as some remarks on the PCT and spin-statistics

theorems for this particular model are given in a final Discussion section (Section 5).

Note that a different approach (in appearance) is adopted by [7], who also obtain a stable, causal QFT based on a preferred frame and a non-standard synchronization scheme, without, however, a discussion of the renormalizability of their theory. Also, they obtain some first results for the beta decay spectrum near the endpoint (for tachyonic neutrinos), as well as suggest an alternative mechanism to neutrino oscillations involving 3-body (tachyonic) decay channels. We conjecture that their QFT approach and ours can be mapped to each other by a suitable (general linear) coordinate transformation. It is hoped that, if these two approaches are indeed found to be compatible, that the present formulation in terms of the more familiar Minkowski coordinates would bring further clarity to the approach of [7].

2 Necessity of Lorentz symmetry breaking

Here we draw upon tools from the microlocal approach to quantum field theory. This approach grew out of the necessity of dealing with the singularities inherent in quantum field theory on curved spacetime [3, 12, 38] in a general and coherent manner. Specifically, one applies techniques and theorems from microlocal analysis [16, 9] to the Green's functions (or, rather, distributions) of the quantum field theory. Such distributions include the advanced and retarded fundamental solutions Δ_A, Δ_R to the inhomogeneous wave equation, the Feynman propagator Δ_F (and its complex conjugate), the two-point distribution $\Delta^{(+)}(x_1, x_2) = \langle 0 \,|\phi(x_1)\phi(x_2)|\, 0\rangle$, and, more generally, the n-point or Wightman distributions, which are the vacuum expectation values of the n-fold products of the field operator $\phi(x)$ at the n points $x_1, \ldots, x_n$. We remark that this incorporation of microlocal techniques into quantum field theory on curved spacetime has led to, among other results, a characterization of the *global Hadamard condition* [23] in terms of a restriction on the *wave front set* of the two-point distribution [29], a resolution of Kay's singularity conjecture [27, 28], and the development of an Epstein-Glaser-like approach to renormalization on curved spacetime

[4, 15]. Note that, as input to the renormalization programme, the two-point distribution must be kept globally Hadamard. Since the characterization of this condition in terms of wave front sets is important in the present context, we start with an introduction to the main concepts involved in this characterization in the following paragraph.

Recall that the *singular support* of a distribution $F(x)$ consists of all the points x at which F is not smooth. (In order for F to be smooth at x, an open neighbourhood of this point must exist, on which F and all its derivatives exist and are finite.) The *wave front set* of F is an extension of the singular support of F consisting of the pairs (x, k) where x is in the singular support, and $k \neq 0$ is a direction in the cotangent space of the spacetime at x. The wave front set is *conic* in the sense that if k is the second component of a point in the wave front set, then so is αk, where $\alpha > 0$ is a real number. The directions k indicate, heuristically speaking, the directions in the "local Fourier space" at x in which the "local Fourier transform" of F near x, along with all its derivatives, does *not* decay more rapidly than any polynomial in the Euclidean distance as this distance tends to infinity. The wave front set is a general enough construct that it can be defined on a curved spacetime as readily as on a flat spacetime, and, on flat spacetime, it is defined for distributions more general than the tempered distributions (whose Fourier transforms exist). Furthermore, some general results for performing operations with such distributions on manifolds (such as multiplication, convolution, and restriction to a submanifold) have simple statements in terms of their wave front sets. For more precise definitions, please see [17].[1]

We now describe the wave front set of a two-point distribution $F^{(+)}(x_1, x_2)$ (on a globally hyperbolic curved spacetime (M, g)) satisfying the Hadamard condition [27, 29].[2] The quadruple (x_1, k_1, x_2, k_2) is in the wave front set precisely when x_1 and x_2 either coincide or

[1] Note that we have implicitly used a definition of Fourier transform that differs from that of [17] by an extra minus sign in the argument of the exponential function. This accounts for the apparent discrepancy in signs in some formulae, e.g., the exponent in our formula Eq.(40) is minus that in the formula for a Fourier integral operator used in Theorem 8.1.9 of [17].

[2] For corrections to the original proofs given in these references (which contain gaps), please consult [24, 32].

are on the same null geodesic. If $x_1 = x_2$, the covectors k_1, k_2 are both null and $k_1 = -k_2$. If $x_1 \neq x_2$ (but x_1 and x_2 are still on the same null geodesic) then k_1, k_2 are null and tangent to this null geodesic at the points x_1, x_2 respectively. Furthermore, k_1 is minus the parallel transport of k_2 from x_2 to x_1 along this null geodesic. An additional restriction is that k_1 is always pointing in the *future* time direction (for both $x_1 = x_2$ and $x_1 \neq x_2$). Thus, k_2 is always pointing in the *past* time direction. There are no other restrictions on the covectors; hence all covectors satisfying the above criteria are included in the wave front set. If, in addition, the spacetime is Minkowski, and translation invariance holds, the two-point distribution is a distribution of the difference variable $x = x_1 - x_2$, and the wave front set of the distribution in one variable $f(x)$ consists of pairs of points (x, k) where x and $k \neq 0$ are null, k (as a vector) is parallel (or anti-parallel) to x, if $x \neq 0$, and k is future pointing.

We sketch the steps of the proof that for a tempered, Poincaré invariant two-point distribution satisfying the tachyonic Klein-Gordon equation $(\Box - m^2)\phi = 0$, the Hadamard condition cannot be satisfied. (Temperedness is assumed here in order to guarantee good behaviour of renormalized observables, e.g., to avoid exponential growth of observables in time.) Translation invariance implies that we can write the two-point distribution as a distribution of the difference variable $x = x_1 - x_2$, i.e., as $u(x)$. Now $\hat{u}(k)$ is also a tempered distribution, and the wave equation implies that $\hat{u}(k)$ is nonzero only for $k^2 = -m^2$, which is a one-sheeted hyperboloid. Furthermore, Lorentz invariance of the distribution $u(x)$ implies Lorentz invariance of $\hat{u}(k)$. The above considerations imply that $\hat{u}(k)$ may be written as $c(k)\delta(k^2 + m^2)$, where $c(k)$ is a smooth function on the one-sheeted hyperboloid. Since a spacelike k may be mapped by a proper, orthochronous Lorentz transformation to $-k$, we obtain $c(k) = c(-k)$ (in fact c is a constant), and therefore $\hat{u}(-k) = \hat{u}(k)$. This implies $u(-x) = u(x)$, which, as is readily verified, leads to the result that $(x, k) \in \mathrm{WF}\ (u)$ if and only if $(-x, -k) \in \mathrm{WF}\ (u)$. Thus the wave front set covectors are not restricted to be future pointing; past pointing ones are also required. (Note that the wave front set of the two-point

distribution must contain *some* pairs, if this theory is to even vaguely resemble a typical nontrivial quantum field theory!) This finishes the sketch that the Hadamard condition cannot be satisfied for a Poincaré invariant, tempered two-point distribution of a tachyonic quantum field theory.

We note that a similar "no-go" result, in which the requirement of temperedness is dropped, appears to have the following counterexample: First note that the symmetric part of the two point distribution (without exponentially growing/decaying modes) is Poincaré invariant, while the same is true of the commutator distribution (determined through the Leray-Lichnerowicz uniqueness theorem, which *requires* the presence of the growing and decaying modes). Thus, we may combine these terms to form a "hybrid" two-point distribution which satisfies the Hadamard condition. However, Wightman positivity evidently fails for this two-point distribution (at least it apparently cannot be constructed from a mode sum). Hence it seems reasonable to conjecture that a Poincaré invariant two-point distribution, satisfying Wightman positivity and the tachyonic Klein-Gordon equation, *cannot* satisfy the Hadamard condition. The author is aware only of arguments in favour of this conjecture being true [11]. In any case, the constructed examples lend support to this conjecture's validity.

A further observation is that, with the insertion of the extra assumption of Wightman positivity for the two-point distribution in our no-go "lemma" for the tempered case, we can assert more strongly that the two-point distribution becomes *real* $u(x)^* = u(x)$, in addition to being even $u(-x) = u(x)$. This is so because Wightman positivity implies that $\hat{u}(k)$ is real and positive-valued (in the appropriate sense of distribution theory), besides being even $\hat{u}(-k) = \hat{u}(k)$. Hence the antisymmetric (and imaginary) part of $u(x)$ is zero. This appears to go far astray from describing a QFT with which we are presently familiar. In any case, the Hadamard condition still fails (this is now readily seen in the fact that the usual leading order Hadamard singularity must have a non-zero imaginary part). As before, the

breakdown of this important input to renormalization forces us to consider such a model as *unphysical* for our purposes.

Finally, we note that non-Hadamard two-point distributions render the existence of the renormalized stress-energy tensor problematic, since the conditions on the wave front set under which one can construct the two-point distribution of the Wick-ordered polynomials (of which the stress-energy tensor is one) are not satisfied. See [5, 36, 37] for more discussion on the stress-energy tensor and the need for the Hadamard condition to be satisfied in order that the stress-energy tensor be defined.

3 Construction of the scalar model

Having argued that one should expect Poincaré symmetry to be broken in a reasonable quantum field model satisfying the tachyonic Klein-Gordon equation, we now present such a model. We shall, in this and the next sections, perform the constructions in just the preferred frame. Note that values of the scalar Green's functions in a boosted frame are readily obtained from these by the obvious change of coordinates. E.g., if a Green's function in the preferred frame is $G(x', y')$, then the same function in the frame obtained by the boost $\Lambda\colon x' \to x$ is the pullback by the inverse Lorentz transformation, namely $G(\Lambda^{-1}x, \Lambda^{-1}y)$.

The first part of the construction in any frame is a listing of all the *oscillatory* mode solutions of the tachyonic Klein-Gordon equation (using $\Box = \partial_t^2 - \nabla^2$)

$$(\Box - m^2)u = 0 \, . \tag{1}$$

In our choice of preferred frame, one has a natural choice of *inner product* for such modes, namely

$$(u, v) = i \int_{t=a} u^*(x) \overleftrightarrow{\partial_t} v(x) \, d^3\mathbf{x} \tag{2}$$

where

$$u^*(x) \overleftrightarrow{\partial_t} v(x) = u^*(x)(\partial_t v(x)) - (\partial_t u^*(x))v(x) \, . \tag{3}$$

This inner product turns out to be independent of the choice of the constant a (as will be evident for the orthonormal basis we shall construct). However, the inner product *does* in general depend on the choice of space-like hypersurface over which the integral is defined. This is in constrast to the case of massive or light-like particles.

We choose (in this reference frame) a basis for the space of *positive energy* or *positive frequency* (oscillatory) solutions as follows. We consider the solutions $u_{\mathbf{k}}(t,\mathbf{x}) = M_{\mathbf{k}}\mathrm{e}^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}$, where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 - m^2}$ and $|\mathbf{k}| > m$. In order to obtain the orthonormal relations

$$(u_{\mathbf{k}}, u_{\mathbf{l}}) = \delta^{(3)}(\mathbf{k}-\mathbf{l}) \tag{4}$$

$$(u^*_{\mathbf{k}}, u^*_{\mathbf{l}}) = -\delta^{(3)}(\mathbf{k}-\mathbf{l}) \tag{5}$$

$$(u_{\mathbf{k}}, u^*_{\mathbf{l}}) = (u^*_{\mathbf{k}}, u_{\mathbf{l}}) = 0 \ , \tag{6}$$

we choose the normalization factors to be $M_{\mathbf{k}} = ((2\pi)^3 \cdot 2\omega_{\mathbf{k}})^{-\frac{1}{2}}$, whence

$$u_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^3 \cdot 2\omega_{\mathbf{k}}}}\mathrm{e}^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})} \ . \tag{7}$$

For mode solutions with $|\mathbf{k}| = m$, which are also oscillatory, the frequency is 0, and thus time derivatives of the modes give zero. Thus the modes here are of the form

$$v_{\mathbf{k}}(t,\mathbf{x}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \tag{8}$$

with

$$(v_{\mathbf{k}}, v_{\mathbf{l}}) = 0 \ . \tag{9}$$

Hence any normalization constant will do here, and we just leave the $v_{\mathbf{k}}$ as they are. Note that inner products with the $u_{\mathbf{k}}, |\mathbf{k}| > m$ are zero. It will turn out that incorporating these modes into the theory does not ultimately affect the two-point distribution, since the integrations involving only these "zero modes" are over a set of measure zero. Hence we may safely ignore them in the further development of the theory.

Note that the *negative energy* or *negative frequency* solutions are simply taken to be the complex conjugates of the positive energy modes. Thus they are

$$u^*_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^3 \cdot 2\omega_{\mathbf{k}}}} \mathrm{e}^{i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} \ , \tag{10}$$

where the parameter space labelled by $\mathbf{k}$ is restricted to $|\mathbf{k}| > m$ as above.

As stated in the Introduction, the exponentially growing and decaying modes, which would be labelled by $|\mathbf{k}| < m$, are omitted from the model in order to avoid exponential blow-up of renormalized observables, and thus they shall be ignored henceforth in this paper. A model in which these are incorporated (as partly described in the Introduction) was given in [33], and also was considered in the context of quantum field theory on curved spacetime by [30, 11].

In order to quantize Eq.(1) without the exponentially growing/decaying modes, we seek a field operator $\phi(x)$ of the following form

$$\phi(x) = \int_{|\mathbf{k}|>m} \left(a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)\right) d^3\mathbf{k} \ , \tag{11}$$

which, besides satisfying Eq.(1), also satisfies the *equal time commutation relations*, modified so as not to include frequencies $\mathbf{k}$ for which $|\mathbf{k}| < m$:

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] \ = \ 0 \tag{12}$$

$$[\phi(t,\mathbf{x}), \partial_t\phi(t,\mathbf{y})] \ = \ i\delta_m(\mathbf{x}-\mathbf{y}) \ . \tag{13}$$

Here, δ_m is a modification of the Dirac delta distribution obtained by deleting modes $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ with frequencies $|\mathbf{k}| < m$ from the latter, namely,

$$\delta_m(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{|\mathbf{k}|>m} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \, d^3\mathbf{k} \ . \tag{14}$$

We find the following relations:

$$(u_{\mathbf{k}}, \phi) = a_{\mathbf{k}}, \ (u^*_{\mathbf{k}}, \phi) = -a^\dagger_{\mathbf{k}} \ , \tag{15}$$

and

$$[a_{\mathbf{k}}, \phi(x)] = u^*_{\mathbf{k}}(x) \ , \tag{16}$$

where we have employed

$$\int u^*_{\mathbf{k}}(t, \mathbf{y}) \delta_m(\mathbf{x} - \mathbf{y}) \, d^3\mathbf{y} = u^*_{\mathbf{k}}(t, \mathbf{x}) \tag{17}$$

for $|\mathbf{k}| > m$. These relations lead to the commutators

$$[a_{\mathbf{k}}, a_{\mathbf{l}}] = \left[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{l}}\right] = 0, \ \left[a_{\mathbf{k}}, a^\dagger_{\mathbf{l}}\right] = \delta^{(3)}(\mathbf{k} - \mathbf{l}) \ . \tag{18}$$

The *vacuum* or *ground state* $|0\rangle$ associated with this particular choice of preferred tachyon frame is then defined by $a_{\mathbf{k}} |0\rangle = 0$. The *two-point distribution* is therefore

$$\begin{aligned} \Delta^{(+)}(x, y) &= \langle 0| \, \phi(x)\phi(y) \, |0\rangle & (19) \\ &= \int_{|\mathbf{k}|>m} u_{\mathbf{k}}(x) u^*_{\mathbf{k}}(y) \, d^3\mathbf{k} & (20) \\ &= \frac{1}{2}\Delta^{(1)}(x, y) + i\frac{1}{2}\Delta(x, y) \ . & (21) \end{aligned}$$

Taking the anti-symmetric part of the two-point distribution, namely,

$$\langle 0| \, [\phi(x), \phi(y)] \, |0\rangle = i\Delta(x, y) = 2i\mathrm{Im} \, \langle 0| \, \phi(x)\phi(y) \, |0\rangle \ \ , \tag{22}$$

we obtain the commutator distribution

$$\Delta(x, y) = -\frac{1}{(2\pi)^3} \int_{|\mathbf{k}|>m} d^3\mathbf{k} \, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{\sin[\sqrt{\mathbf{k}^2 - m^2}(t - s)]}{\sqrt{\mathbf{k}^2 - m^2}} \ . \tag{23}$$

This is seen to be the unique distributional bisolution of the Cauchy problem

$$\begin{aligned} \Delta(t, \mathbf{x}, t, \mathbf{y}) &= 0 & (24) \\ \partial_t \Delta(t, \mathbf{x}, s, \mathbf{y})|_{s=t} &= -\delta_m(\mathbf{x} - \mathbf{y}) \ . & (25) \end{aligned}$$

The symmetric part of the two-point distribution, namely

$$\Delta^{(1)}(x, y) = \langle 0| \, \{\phi(x), \phi(y)\} \, |0\rangle = 2\mathrm{Re} \, \langle 0| \, \phi(x)\phi(y) \, |0\rangle \ \ , \tag{26}$$

is

$$\Delta^{(1)}(x,y) = \frac{1}{(2\pi)^3}\int_{|\mathbf{k}|>m} d^3\mathbf{k}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{\cos[\sqrt{\mathbf{k}^2-m^2}(t-s)]}{\sqrt{\mathbf{k}^2-m^2}} \; . \tag{27}$$

Note that this is a Lorentz invariant distribution.

4 Renormalized operators

Taking the (classical) Lagrangian density of our field theory to be

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2\right) \; , \tag{28}$$

where the field is real-valued, we obtain, via the usual method, the Hamiltonian

$$H = \frac{1}{2}\int \left(\pi^2 + (\nabla\phi)^2 - m^2\phi^2\right) \, d^3\mathbf{x} \; , \tag{29}$$

where $\pi(x) = \partial_t\phi(x)$. Note that, because of the minus sign in the third term above, the Hamiltonian is generally not positive. However, we shall shortly show that the quantized version is indeed a positive operator in the preferred frame. Similarly, the momentum of the field is

$$\mathbf{P} = -\int \partial_t\phi\nabla\phi \, d^3\mathbf{x} \; . \tag{30}$$

The quantization of the above expressions proceeds along the familiar lines: classical fields are replaced by the quantized versions, and products of field operators are normal ordered. We express the final result in terms of creation/annihilation operators, recalling that normal ordering places creation operators before annihilation operators in the expressions. Thus, we obtain

$$\pi \;=\; \partial_t\phi = -i\int_{|\mathbf{k}|>m} \omega_{\mathbf{k}}(a_{\mathbf{k}}u_{\mathbf{k}} - a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}})\, d^3\mathbf{k} \tag{31}$$

$$\nabla\phi \;=\; i\int_{|\mathbf{k}|>m} \mathbf{k}(a_{\mathbf{k}}u_{\mathbf{k}} - a^\dagger_{\mathbf{k}}u^*_{\mathbf{k}})\, d^3\mathbf{k} \; , \tag{32}$$

and, using the orthonormality property of the exponential functions $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$,

$$
\begin{aligned}
H &= \int_{|\mathbf{k}|>m} \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\, d^3\mathbf{k} && (33)\\
\mathbf{P} &= \int_{|\mathbf{k}|>m} \mathbf{k} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\, d^3\mathbf{k}\ . && (34)
\end{aligned}
$$

Note that these expressions are as expected, in analogy with the usual creation/annihilation operator formalism in conventional QFT. We anticipate that these operators will transform in the usual manner when we calculate them in a boosted frame. Hence the Hamiltonian, although it will remain Hermitian, will not remain positive in any frame boosted with respect to the preferred one.

For the case of a charged scalar tachyon, the Lagrangian density becomes

$$
\mathcal{L} = \partial^{\mu}\phi^{*}\partial_{\mu}\phi + m^2\phi^{*}\phi\ , \tag{35}
$$

where the field is now complex-valued. The conserved quantity derived from global phase invariance (i.e., invariance under $\phi \to \mathrm{e}^{i\alpha}\phi$) is

$$
Q = -iq\int \phi^{*}\overleftrightarrow{\partial_t}\,\phi\, d^3\mathbf{x}\ . \tag{36}
$$

The ansatz for the field operator is now

$$
\phi = \int_{|\mathbf{k}|>m} (a_{\mathbf{k}}u_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*})\, d^3\mathbf{k}\ , \tag{37}
$$

while the quantized version of the above conserved quantity evaluates to

$$
\begin{aligned}
Q &= -iq\int :\phi^{\dagger}\overleftrightarrow{\partial_t}\,\phi:\ d^3\mathbf{x} && (38)\\
&= q\int_{|\mathbf{k}|>m} (b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}} - a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}})\, d^3\mathbf{k}\ . && (39)
\end{aligned}
$$

Thus the operator $b_{\mathbf{k}}^{\dagger}$ may be interpreted as creating a particle of charge q from the vacuum, while $a_{\mathbf{k}}^{\dagger}$ may be interpreted as creating an anti-particle of charge $-q$ from the vacuum.

5 Discussion

A sketch that the two-point distribution constructed in Section 3 is of the Hadamard form is now presented. One notes that the two-point distribution (of the difference variable) for our tachyonic theory may be written as

$$\Delta^{(+)}(x) = \frac{1}{(2\pi)^3} \int_{|\mathbf{k}|>m} \frac{1}{2\omega_{\mathbf{k}}} \mathrm{e}^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})}\, d^3\mathbf{k} \; . \tag{40}$$

It is simple to show that the *scaling limit* of our two-point distribution is

$$\lim_{\lambda\to 0} \lambda^2 \Delta^{(+)}(\lambda x) = \Delta_0^{(+)}(x) \; , \tag{41}$$

which is the two-point distribution for the massless theory. Thus the leading order singularity of $\Delta^{(+)}(x)$ is the usual Hadamard one. Conceivably, $\Delta^{(+)}(x)$ may have lower order singularities different from the ones emanating along the light cone from the origin, or the lower order singularities may produce extra covectors for the singularities emanating from the origin. However, noting that Eq.(40) describes a *Fourier integral operator* [16], with *homogeneous phase function* $\phi(x, \mathbf{k}) = |\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x}$, we may avail ourselves of Theorem 8.1.9 of [17],[3] which restricts the pairs (x, k) in the wave front set to be precisely those in the leading order singularity. Thus the wave front set of $\Delta^{(+)}$ is of the Hadamard type. Since the anti-symmetric part of $\Delta^{(+)}$ is the same as i times the advanced minus retarded fundamental solutions, up to smooth function, the equivalence theorem of [29] tells us that $\Delta^{(+)}$ satisfies the global Hadamard condition. Thus, we expect that the free field theory presented in Section 3 may be used as input to a renormalizable (self-) interacting theory, satisfying the criteria of Weinberg's theorem [4]. That it leads to sensible renormalized expectation values of observables quadratic in the fields has already been demonstrated for some typical cases in Section 4.

We now explain the "no anti-telephones" property of this model, which we consider to be a version of *chronology protection*, as mentioned in the Introduction. An *anti-telephone*

[3] See Footnote 1.

is a device in which a relay at B is set up at a spacelike separation to a message sender at A. The relay is designed to receive a message sent via tachyons from A and immediately to resend the message, again using tachyons, back to a receiver at A' (the same individual as the initial sender, but at a different time). The idea of the anti-telephone is to attempt to violate causality by sending the relayed message from B to A' *more backward in time* (resp. less forward in time) than the starting message was sent from A to B forward in time (resp. backward in time). Then one would have used tachyons to effect what should be considered a highly egregious violation of causality. (If this could be done, then a signal could be sent whose effect would be to prevent the message from being sent in the first place. However, if the message had not been sent, then nothing would have hindered it from being sent.) However, it is apparent that, due to the cutoff in the spectrum of the one-particle states of the model presented in Section 3, there is a global directional dependence in the lower bound of the allowed energies, such that, if the tachyons managed to travel backward in time in moving from A to B, they would certainly be forced to travel as much forward in time, or more so, in travelling back from B to A'. Only particles with such a directionally dependent lower bound in the energy may be created out of the vacuum either in the preferred frame, or in any other frame, boosted with respect to the preferred one. Similarly, no sequence of relays could be constructed to guide the tachyons along some path so that they arrive back to A' at a previous time to A, since it is clear that such a path of tachyonic world-lines, any one of which goes backward in time, is not constructible in the preferred frame. If it is impossible in one inertial frame, then it must be so for all inertial frames. Note that we should postulate that the same preferred frame must be universal to all tachyonic particles in the same interacting theory, since otherwise, severe violations of causality, in principle, could be brought about.

It is of interest to determine whether any of the "big theorems" of axiomatic QFT [34, 19] remain in our model, or in models constructed in a similar approach (by restricting 4-

momenta to lie in the upper half of a single-sheeted hyperboloid cut through by a spacelike hyperplane through the origin). As a first step, we consider the PCT theorem for the Hermitian scalar model. The relevant property to be proved for the two-point distribution is

$$\Delta^{(+)}(x)^* = \Delta^{(+)}(-x) \ . \tag{42}$$

This is equivalent to the statement that the Fourier transform of $\Delta^{(+)}$ is real-valued, which is evidently true for our model in any inertial frame. (The Fourier transform of the two-point distribution is a positive multiple of $\theta(k^0+\beta k^z)\delta(k^2+m^2)$ in a frame boosted by a speed β in the z direction relative to the preferred frame.) Hence a PCT theorem holds for this model. (The above property, appropriately reformulated, extends to all the Wightman distributions, since the theory we have constructed is *quasi-free*, and the Wightman distributions determine the full theory by the Wightman reconstruction theorem [39].)

Next, we touch upon the spin-statistics theorem. We would expect that a well-behaved tachyonic model would reduce to a physically well-behaved massless theory as the tachyonic parameter m tends to 0. That would mean that the Gårding-Wightman axioms (Wightman positivity, Poincaré invariance, spectral condition, local commutativity) should hold for the scaling limit two-point distribution. However, if the wrong connection between spin and statistics is assumed in this case, then one would necessarily obtain a non-Lorentz invariant theory in the scaling limit, since all the other properties would presumably be satisfied for this limit. (If all the axioms hold in the limit, the limit two-point distribution must be zero, by the usual spin-statistics theorem. However, this would contradict the definition of the scaling limit as the leading order [non-zero] behaviour of the two-point distribution as the difference variable x tends to 0, unless, of course, the two-point distribution of the original tachyonic theory is itself 0.) Thus, to avoid this undesirable failure of Lorentz invariance in the scaling limit, we must retain the usual connection between spin and statistics. Note that this model then stands in constrast to the one suggested by Feinberg [10], who assumed

the wrong connection of spin with statistics, e.g., anti-commutation relations in the scalar theory.

Finally, we observe that the spacelike hypersurface through the origin (in Fourier space), which bounds the upper half of the single-sheeted mass hyperboloid from below (i.e., the one-particle spectrum of the model described here), may be regarded as defining a *frame-dependent interpretation rule* for the allowed 4-momenta of particles and anti-particles in the QFT. This is, in effect, a use of the "Re-interpretation Principle" of [2], which proposes to regard a negative energy, backward-in-time-moving particle/anti-particle of momentum $\mathbf{k}$ as a positive energy, forward-in-time-moving anti-particle/particle of momentum $-\mathbf{k}$. This would at first seem to suggest an identification of the 4-momentum k with $-k$ on the full single-sheeted hyperboloid. However, we find it more appropriate to pick a *single* description from each pair $(k, -k)$, to describe *both* a particle and anti-particle, and to do so in each frame in a way that preserves chronology protection (a plane must be used to cut the hyperboloid) and the Hadamard condition (the half containing arbitrarily large *positive* energies must be chosen), and is consistent with Lorentz covariance. (The fact that, in the quantum field model, we choose a single description from among two descriptions which are equally valid from the classical viewpoint, suggests an *economy of description* principle inherent in the quantum field theory of tachyons. This, of course, is satisfied in the usual choice of the upper mass hyperboloid [out of two sheets], as is made in the regular massive Klein-Gordon theory.) Such a "halving" of the single sheeted mass hyperboloid is, up to a boost, unique. We find it rather remarkable that such a simple interpretaion rule leads to *both* chronology protection *and* the Hadamard condition (i.e., renormalizability) being satisfied. This points to a deep unity among the "axioms" which we adopt as physical.

In conclusion, we expect that future work in this subject will be done to further develop and clarify the quantum field theoretic aspects of tachyons, especially those involving interactions. A basic step in this direction has been to clarify the calculation of the phase space

factor that appears in two-body decay in which one of the products is a tachyon [31]. Note that, in that paper, the underlying quantum field theory is implicitly assumed to be in accord with the model presented in this paper (in Section 3). We also foresee the development started here as extending consistently to Dirac-like tachyonic (*Dirachyonic*) quantum field theory, whose ramifications (especially the inherent maximal parity breaking that arises in such a model) would tend to support the possibility that the neutrino may be a tachyon.

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